One third of poset combinatorics was missing

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Outline

Loeb's binomials

Poset combinatorics

Shuffles

This talk is based on the preprint "One third of poset combinatorics was missing", that we aim to make public after February 2015.
Joint work with Khushdil Ahmad, Sangil Kim, and Khurram Shabbir.

Binomials

Loeb introduced three binomial coefficients (Loeb [1992b]).

Value	Range of parameters
$\binom{n}{k}$	$n \geq k \geq 0$
$(-1)^k \left(\!\binom{-n}{k}\!\right)$	$k \geq 0 > n$
$(-1)^{n+k} \left(\!\binom{-n}{-(k-n)}\!\right)$	$0 > n \geq k$

Here $\left(\!\binom{n}{k}\!\right) := \binom{n+k-1}{k}$.

Loeb's yoga:

A single proof of a binomial identity corresponds to three distinct identities if we use certain axioms.
Example:

Theorem (Anelli et al. [1995])

For all integers i, j, k,

$$\binom{i}{j}\binom{j}{k} = \binom{i}{k}\binom{i-j}{j-k}.$$

Metamathematics

Yoga (Combinatorial multiverse)

If your combinatorial problem determines an expression of the form

$$\sum_{i=n_0}^{n_1} c_i \binom{x}{i}, c_i \in \mathbb{Z}$$

for one interpretation of the binomial coefficient, then

- (Loeb) there are three formulas,

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for one interpretation of the binomial coefficient, then

- (Loeb) there are three formulas,
- there are three versions of your combinatorial problem.

The multiverse in media

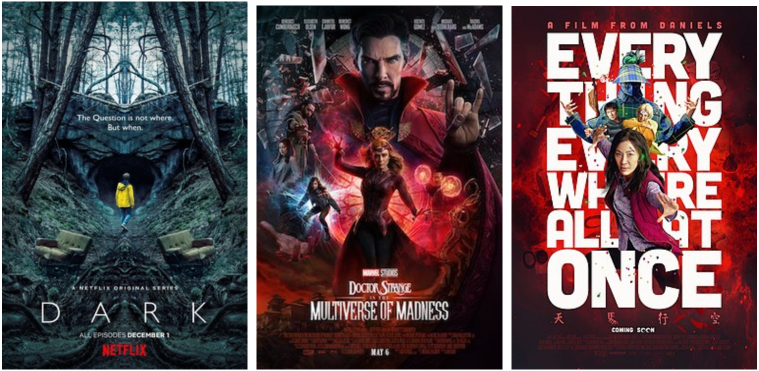

Figure: At least one of this works is related to the multiverse

Order preserving maps in the multiverse

Denote by $\langle n \rangle = \{1 < 2 < \cdots < n\}$ the linear order on n points, deck of n cards.

- $\binom{n}{k} = \#\{f : \langle k \rangle \to \langle n \rangle |$ if $x < y$ then $f(x) < f(y)\}$ (strict order preserving),
  Example,
  $$\{f : \langle 2 \rangle \mapsto \langle 3 \rangle\} :$$
  - $(1,2) \to (1,2)$
  - $(1,2) \to (1,3)$
  - $(1,2) \to (2,3)$

Order preserving maps

- $\left(\!\binom{n}{k}\!\right) = \#\{f : \langle k \rangle \to \langle n \rangle |$ if $x < y$ then $f(x) \leq f(y)\}$
  (weak order preserving),
  Example,
  $$\{f : \langle 2 \rangle \mapsto \langle 3 \rangle\} :$$
  - $(1,2) \to (1,2) \qquad (1,2) \to (1,1) \qquad (1,2) \to (3,3)$
  - $(1,2) \to (1,3) \qquad (1,2) \to (2,2) \qquad (1,2) \to (2,3)$

Order preserving maps

- $\left(\!\!\binom{n}{k-n}\!\!\right) = \#\{f : \langle k\rangle \twoheadrightarrow \langle n\rangle |$ if $x < y$ then $f(x) \leq f(y)\}$
  (weak surjective order preserving)
  Example,
  $$\{f : \langle 3\rangle \mapsto \langle 2\rangle\} :$$
  - $(1, 2, 3) \to (1, 1, 2)$
  - $(1, 2, 3) \to (1, 2, 2)$

Posets

A poset is a set with a partial order.
We represent posets with Hasse diagrams where $y = \text{succesor}(x)$ is represented by a vertical vertex from x (below) to y:

$$\{w, x, y, z | x < y < z\} =$$

Poset combinatorics

Definition (Stanley [1970])

The strict order polynomial $\Omega^{\circ}(P)(n) := \#hom_{\text{strict}}(P, < n >)$.

For example, $\Omega^{\circ}($  $)(x) := 2\binom{x}{3} + \binom{x}{2}$.
When $x = 3$ we obtain $2(1) + 3 = 5$.

Stanley order polynomials

Definition (Stanley [1970])

The weak order polynomial $\Omega(P)(n) := \#hom_{\text{weak}}(P, < n >)$.

For example, $\Omega($$)(x) := 2\left(\!\!\binom{x}{3}\!\!\right) - \left(\!\!\binom{x}{2}\!\!\right) = 2\binom{x+3-1}{3} - \binom{x+2-1}{2}$.
When $x = 2$ we obtain $2(4) - 3 = 5$.

Stanley order polynomials

Given a finite poset P, we have already order polynomials for two 'Universes':

$$\Omega^{\circ}(P)(x) = \sum_{i=1}^{|P|} d_{P,i} \binom{x}{i}, \quad \Omega(P)(x) = \sum_{i=1}^{|P|} (-1)^{|P|-i} d_{P,i} \left(\!\binom{x}{i}\!\right).$$

Question:

Is there a third order polynomial obtained using $\left(\!\!\binom{x}{k-x}\!\!\right)$?

Unfortunately, $\left(\!\!\binom{x}{k-x}\!\!\right) = \binom{k-1}{x-1}$ is not a polynomial on x.

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Since:

$$\sum_{m=1}^{i} (-1)^{m-i} \left(\!\!\binom{m}{i-m}\!\!\right) x^m = x(x-1)^{i-1},$$

We can work with generating series. The generating series of $\binom{x}{i}$ is $\sum \binom{n}{i} x^n = \frac{x^i}{(1-x)^{i+1}}$, and the generating series of $\left(\!\!\binom{x}{i}\!\!\right)$ is $\frac{x}{(1-x)^i}$.

Order series in the multiverse

Definition

Given a poset P, we define *for* $\binom{n}{k}$:

$$\mathfrak{Z}(P) := \sum_{n=0}^{\infty} \Omega^{\circ}(P,n)x^n = \sum_{i=1}^{|P|} d_{P,i} \frac{x^i}{(1-x)^{i+1}}$$

for $\left(\!\binom{n}{k}\!\right)$:

$$\mathfrak{Z}^{+}(P) := \sum_{n=0}^{\infty} \Omega(P,n)x^n = \sum_{i=1}^{|P|} (-1)^{|P|-i} d_{P,i} \frac{x}{(1-x)^{i+1}},$$

for $\left(\!\binom{n}{k-n}\!\right)$:

$$\mathfrak{Z}^{s}(P) := \sum_{i=1}^{|P|} (-1)^{|P|-i} d_{P,i} x(1-x)^{i-1}$$

Order series in the multiverse

Remember that $\left(\!\binom{n}{k-n}\!\right)$ counts weak surjective order preserving maps from $\langle k\rangle$ to $\langle n\rangle$.

Theorem (Ahmad et al. [2023])

Let P be a finite poset. Then,

$$\mathfrak{Z}^s(P) = \sum_{n=0}^{\infty}(-1)^{|P|-n}\#\operatorname{hom}_{s,w}(P,\langle n\rangle)x^n,$$

where $\operatorname{hom}_{s,w}(P,\langle n\rangle)$ are weak surjective order preserving maps.

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We started with two order series, and due to the existence of three binomial coefficients we formally found another series *and we found a combinatorial interpretation to the series*.

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The endomorphisms are also isomorphic.

$End_{\{\mathfrak{Z}^+(P)|P\in \text{poset}\}}$, $End_{\{\mathfrak{Z}(P)|P\in \text{poset}\}}$, $End_{\{\mathfrak{Z}^s(P)|P\in \text{poset}\}}$.

Combinatorics of posets before the 32nd Kias combinatorics workshop.

Table: Combinatorics related to binomial coefficients

Object vs Regions	$\binom{n}{k}, n \geq k \geq 0$	$(-1)^k \left(\!\binom{-n}{k}\!\right), k \geq 0 > n$
# subsets of size k out of a set of size n	subsets of a set (?)	submultiset of a multiset (Bhāskarāchārya [1150])
# maps from a poset to a linear order	strict maps ([Stanley, 1970, Theorem 1])	weak maps ([Stanley, 1970, Theorem 3])
# lattice points	Ehrhart series of a polytope (Ehrhart [1962])	Ehrhart series of the interior of a polytope (Ehrhart [1962], Macdonald [1971])
# shuffles between two linear orders	right deck-divider shuffles (Muir [1902])	riffle shuffles (?)

Quote

Our thinking is shaped by dualities.
Black and white...
Light and shadow...
But this is false.
You need a third dimension to fulfill it all.

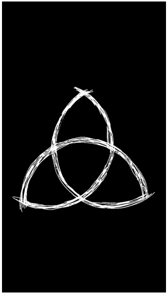

Figure: Dark, Session 3, episode 8.

Poset combinatorics

Table: The three regions of poset combinatorics

Object vs Regions	$\binom{n}{k}, n \geq k \geq 0$	$(-1)^k \left(\!\binom{-n}{k}\!\right), k \geq 0 > n$	$(-1)^{n+k} \left(\!\binom{-n}{-(k-n)}\!\right), 0 > n \geq k$
# subsets of size k out of a set of size n	subsets of a set (?)	submultiset of a multiset (Bhāskarāchārya [1150])	hybrid subsets of a new set ([Loeb, 1992a, Theorem 5.2])
# maps from a poset to a linear order	strict maps ([Stanley, 1970, Theorem 1])	weak maps ([Stanley, 1970, Theorem 3])	weak surjective maps ([Ahmad et al., 2023, Lemma 9])
# lattice points	Ehrhart series of a polytope (Ehrhart [1962])	Ehrhart series of the interior of a polytope (Ehrhart [1962], Macdonald [1971])	$\#\mathcal{L}_{k,n} \cap nPoly(P)$ (Ahmad et al. [2023], Equation (5))
# shuffles between two linear orders	right deck-divider shuffles (Muir [1902])	riffle shuffles (?)	left deck-divider shuffles ([Ahmad et al., 2023, Lemma 12])
# shuffles between a poset and a linear order	right deck-divider shuffles ([Ahmad et al., 2023, Theorem 3])	colimit-indexing shuffles ([Ahmad et al., 2023, Theorem 1])	lelft deck-divider shuffles ([Ahmad et al., 2023, Lemma 12])

Shuffles

The set of (riffle) shuffles of *linear orders* $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_m$ consist of those linear orders $<_{sh}$ on the set $\{a_i\}_{1 \leq i \leq n} \cup \{b_j\}_{1 \leq j \leq m}$, that satisfy:

$$a_1 <_{sh} a_2 <_{sh} \cdots <_{sh} a_n,$$

and

$$b_1 <_{sh} b_2 <_{sh} \cdots <_{sh} b_m.$$

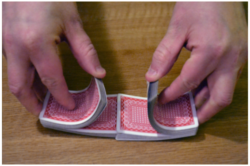

Figure: Image from Hannibal, CC BY-SA 3.0 `https://creativecommons.org/licenses/by-sa/3.0`, via Wikimedia Commons

Shuffles in the region $k \geq 0 > n$

Given two decks, one with $m = -n$ cards and one with k cards, the number of shuffles between them is $\binom{m+k}{k} = \left(\!\binom{m+1}{k}\!\right)$ (a multiset!).

The multiverse

Like Galileo, peering through his telescope at the moons of Jupiter and inferring the existence of other words, catching a glimpse of what it would be like to live on them...
Joel Hamkins, The set-theoretic multiverse (Hamkins [2012]).

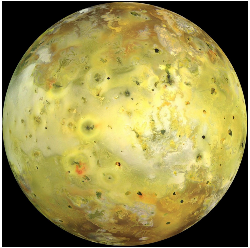

Figure: Io, moon of Jupiter with true colors. Image received from NASA's Galileo spacecraft.

The three Shuffles of decks

Given two decks, one with n cards and one with k cards, a shuffle between them is

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Given two decks, one with n cards and one with k cards, a shuffle between them is

- in the region $k \geq 0 > n$, a riffle shuffle of cards.
- in the region $n \geq k \geq 0$, a shuffle of cards where the first and the last cards are form the $\boldsymbol{n}$ deck and there are not two cards together from the $\boldsymbol{k}$ deck (Muir [1902]).

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- in the region $n \geq k \geq 0$, a shuffle of cards where the first and the last cards are form the $\boldsymbol{n}$ deck and there are not two cards together from the $\boldsymbol{k}$ deck (Muir [1902]).
- in the region $0 > n \geq k$, a shuffle of cards where the first and the last cards are form the $\boldsymbol{k}$ deck and there are not two cards together from the $\boldsymbol{n}$ deck.

The three Shuffles of decks

In which universe do we live? One where a shuffle is

- a riffle shuffle of cards (in the region $k \geq 0 > n$).
- a shuffle of cards where the first and the last cards are form the n deck and there are not two cards together from the k deck (in the region $n \geq k \geq 0$).
- a shuffle of cards where the first and the last cards are form the k deck and there are not two cards together from the n deck (in the region $0 > n \geq k$).

Where do we live?

If we live in a universe where $k \geq 0 > n$, then the restrictions

$$k \geq 0 > -k > n$$

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(compare with $0 > n \geq k$) allow us to study the two other universes.
Perhaps, asking $n \geq k$ when considering subsets, is also a restriction! In our universe ($k \geq 0 > n$) we can always take subsets with replacement.

Is this useful?

How does the knowledge of the combinatorial multiverse can be used in math?

Series parallel poset

A series parallel poset (SP-poset) is a finite poset that does not include the poset $\{x < y > w < z\}$.

Shuffles of a poset and a deck of cards

Definition (right deck-divider shuffle)

Let P be a SP-poset and fix a chain $\langle n \rangle$. Consider the Hasse diagram of the poset P and draw n horizontal lines in such a way that before the first line and after the last line there is at least one point of the poset P, and the lines are not consecutive.

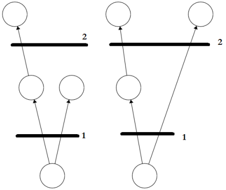

Figure: Two examples of right deck-divider shuffles between a poset  and $\langle 2 \rangle$

Shuffle series

Theorem (Ahmad et al. [2023])

Let P be a series parallel poset, then the series $dd\mathcal{SH}(P) = \sum_{i=1}^{|P|} (-1)^{|P|-i} d_{P,i} (1-x)^{i-1}$, enumerates right deck-divider shuffles between the poset P and chains.

There should be three shuffle series...

We define three sets of series related to shuffles between a SP-poset and a chain.

- The right deck-divider shuffle series $dd\mathcal{SH}(P) = \sum_{i=1}^{|P|}(-1)^{|P|-i} d_{P,i}(1-x)^{i-1}$, this corresponds to the intepretation $\binom{m}{n} = (-1)^{n+m}\left(\!\binom{m}{n-m}\!\right)$.

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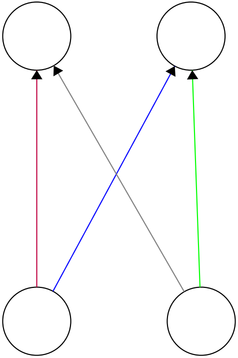
- The colimit-indexing shuffles $\mathcal{SH}(P) = \sum_{i=1}^{|P|} (-1)^{n-i} d_{P,i} \frac{1}{(1-x)^{i+1}}$, this corresponds to the interpretation $\binom{m}{n} = (-1)^{n} \left(\!\binom{m}{n}\!\right)$.
- The left deck-divider shuffle series $ldd\mathcal{SH}(P) = \sum_{i=1}^{|P|} d_{P,i} \frac{x^{i+1}}{(1-x)^{i+1}}$, this corresponds to the usual $\binom{m}{n}$.

Maximal chains

The poset $\{a < c, b < c, a < d, b < d\}$ has 4 maximal chains:



Colimit-indexing shuffle

Given a series parallel poset P, a *colimit-indexing shuffle* between P and $\langle n \rangle$ are those posets A with points labeled by P or $\langle n \rangle$,

- $\{$maximal chains of P$\}$ are isomorphic to $\{$maximal chains of $A\}$.

Colimit-indexing shuffle

Given a series parallel poset P, a *colimit-indexing shuffle* between P and $\langle n \rangle$ are those posets A with points labeled by P or $\langle n \rangle$,

- {maximal chains of P} are isomorphic to {maximal chains of A}.
- To every maximal chain M in P, the isomorphic maximal chain in A is a shuffle of linear orders M and $\langle n \rangle$.

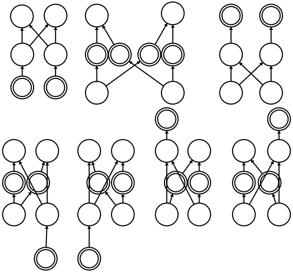


Figure:

Colimit-indexing shuffle between  and $\langle 1 \rangle$.

Shuffle series

Theorem (Ahmad et al. [2023])

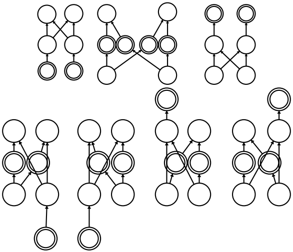
Let P be a series parallel poset, then the series $\sum_{i=1}^{|P|}(-1)^{n-i} d_{P,i} \frac{1}{(1-x)^{i+1}}$ enumerates colimit-indexing shuffles between the poset P and chains.

This answers a question of Hoffbeck and Moerdijk, on enumerating the terms of the tensor product of a tree and a line (operads) in dendroidal homotopy theory.

Example

Let $P = \{a < c, b < c, a < d, b < d\} =$ ,

$$\begin{aligned}
\mathcal{SH}(P) &= 4\frac{1}{(1-x)^5} - 4\frac{1}{(1-x)^4} + \frac{1}{(1-x)^3} \\
&= 1 + (4\binom{4+1}{4} - 4\binom{3+1}{3} + \binom{2+1}{2})x^1 \\
&\quad +(4\binom{4+2}{4} - 4\binom{3+2}{3} + \binom{2+2}{2})x^2 + \cdots \\
&= 1 + 7x^1 + 26x^2 + \cdots
\end{aligned}$$



Definition (left deck-divider shuffle)

Given a series parallel poset P, a left deck-divider shuffle between P and $\langle n \rangle$ is a colimit-indexing shuffle A in which for every maximal chain $m \subset A$, the maximum and minimum points of m are points from $\langle n \rangle$, and there are no two consecutive points of P on m.

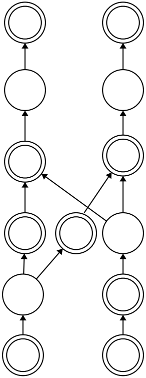


Figure: A left deck-divider shuffle between  and $\langle 4 \rangle$. The elements of the chain are shown with double circles.

Poset combinatorics

Table: The three regions of poset combinatorics

Object vs Regions	$n \geq k \geq 0$	$k \geq 0 > n$	$0 > n \geq k$
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# shuffles between two linear orders	right deck-divider shuffles (Muir [1902])	riffle shuffles (?)	left deck-divider shuffles ([Ahmad et al., 2023, Lemma 12])
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Metamathematics

Yoga: The combinatorial multiverse

If the solution to your combinatorial problem can be written as one of the generating series:

- $\sum_{i=s}^{n} c_i x^r \frac{x^i}{(1-x)^{i+1}}$,
- $\sum_{i=s}^{n} c_i x^r \frac{x}{(1-x)^{i+1}}$,
- or $\sum_{i=s}^{n} c_i x^r x(1-x)^{i-1}$, $c_i \in \mathbb{Z}$,

for $n, r, s \in \mathbb{N}$, then:

- the other formulas correspond to different regions of the binomial parameters,
- and there are three versions of your combinatorial problem.

Gracias I

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