A_∞-ALGEBRAS AND MASSEY PRODUCTS.

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1. ASSOCIATIVITY UP TO HOMOTOPY

Given a binary operation $(\cdot,\cdot)$, we say that it is associative if $(\cdot,(\cdot,\cdot)) = ((\cdot,\cdot),\cdot)$. Sometimes we don't have an associative operation but instead a invertible map

$$\phi : (\cdot,(\cdot,\cdot)) \to ((\cdot,\cdot),\cdot).$$

Using ϕ there are two ways to go from $(\cdot,(\cdot,(\cdot,\cdot)))$ to $(((\cdot,\cdot),\cdot),\cdot)$, see figure 1. When those compositions coincide, and if we keep track on the parenthesization, then this product is well defined[1] as far as we remember to use ϕ any time we want to change the order of composition.

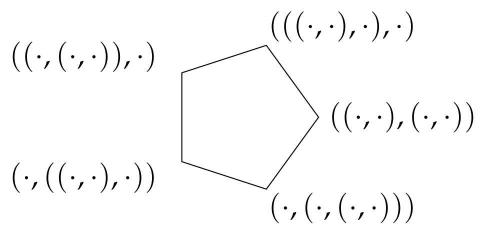


FIGURE 1. Vertices of the Stasheff polytope K^2 are given by ways to compose $(\cdot,\cdot)$ with itself 3 times. Edges are given by ways to apply ϕ.

Today we will see another interesting property of K^2, it's cochain complex has the structure of an associative algebra (up to homotopy). For the cellular decomposition in figure 2 we define a binary operation m_2 on cochains by the rules in figure 3, which should remind us of the usual cup product [2]. For example, the edge w has source b and target d, then

$$m_2(b^*,w^*) = m_2(w^*,d^*) = w^*,$$

while

$$m_2(b^*,d^*) = 0, m_2(w^*,b^*) = m_2(d^*,w^*) = 0.$$

[1] Meaning that starting from $(((\cdots((\cdot,\cdot),\cdot),\cdots),\cdot)$ with an arbitrary number of compositions, we can use iterations of ϕ to end up with a position like $(\cdot,(\cdots(\cdot,(\cdot,\cdot))\cdots)))$ and this is independent of all possible choices in the iteration of ϕ.

[2] Figures 2 and 3 were taken from [1].

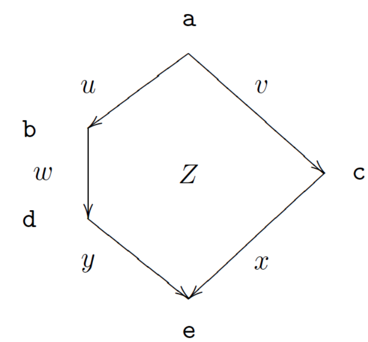


FIGURE 2. Cellular decomposition of K^2.

$\alpha\backslash\beta$	a^*	b^*	c^*	d^*	e^*	u^*	v^*	w^*	x^*	y^*	Z^*
a^*	a^*					u^*	v^*				Z^*
b^*		b^*						w^*			
c^*			c^*						x^*		
d^*				d^*						y^*	
e^*					e^*						
u^*		u^*						Z^*		Z^*	
v^*			v^*						$-Z^*$		
w^*				w^*						Z^*	
x^*					x^*						
y^*					y^*						
Z^*					Z^*						

FIGURE 3. Definition of m_2, all empty entries are cero.

Notice that

$$m_2(m_2(u^*, b^*), y^*) = Z^*,$$

while

$$m_2(u^*, m_2(b^*, y^*)) = m_2(u^*, 0) = 0.$$

So this product is non associative. The boundary map d of chains induces a map d^* on cochains (for example $dw = b - d$ implies that $d^*b^* = u^* - w^*$), and if we define m_3 by

$$m_3(u^*, w^*, y^*) = Z^*,$$

and $m_3(*, *_2, *_3) = 0$ for any other entries, then we can check that equation (1) holds.

$$d^* m_3(id, id, id) + m_3(d^*, id, id) + m_3(id, d^*, id) + m_3(id, id, d^*) =$$
$$= m_2(m_2(id, id), id) - m_2(id, m_2(id, id)), \tag{1}$$

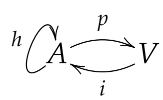
Remark 1.1. *For an n-ary operation $m_n, n > 1$ we define*

$$\partial m_n = dm_n + (-1)^{n-1} \sum m_n(id, \cdots, id, d, id, \cdots).$$

Then (1) *looks like* $\partial m_3 = m_2(m_2(id, id), id) - m_2(id, m_2(id, id))$.

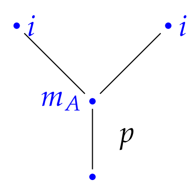
We conclude m_2 is not an associative product, but there is a homotopy m_3 that corrects its failure to be associative.

1.0.1. *Massey products.* We will introduce another example where the equation (1) naturally appears. Lets consider a differential graded associative (DGA) algebra (A, d_A) and a complex (V, d_V) that is a homotopy retract of A :

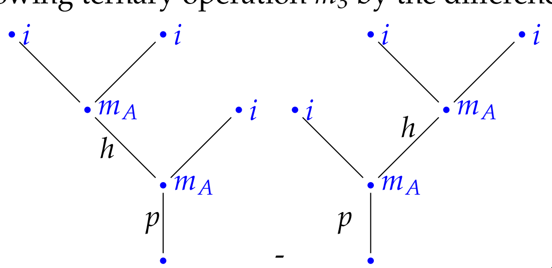


with $Id - ip = dh + hd$. The question that we will answer is: can we induce any algebraic structure on the complex (V, d_V)? We will work over a field k, and it may be convenient to keep in mind the example when the complex V is $(H_*(A), d = 0)$.

A natural idea is to take two elements $a, b \in V$, then map them to A, multiply their images with the operation m_A of A and then send it back to V. We represent this operation m_2 (defined on V!) by a binary tree:



We cannot expect this new structure to be associative, but if we define the following ternary operation m_3 by the difference of trees:

$m_3 =$ 

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then we can prove that $\partial m_3 = m_2(m_2, id) - m_2(id, m_2)$ (see remark 1.1). So we end up with the same equation that in the previous example. When

$V = H_*(A)$, then we are claiming that there is a mysterious ternary operation in cohomology. If we take a, b, c cocycles so that

$$(-1)^{|a|} m_2(a,b) = dX, \quad (-1)^{|b|} m_2(b,c) = dY,$$

then we can find $X,\ Y$ so that $hd(X) = X,\ \ hd(Y) = Y$. After this choices

$$-m_3(a,b,c) = -p(m_2(h(m_2(a,b)),c)) + p(m_2(a,h(m_2(b,c)))),$$

reduces to

$$(-1)^{|a|+|b|}(m_2(a,Y) - m_2(X,c)),$$

so m_3 give us a model for the Massey product.

Remark 1.2. *The Massey product is defined only for cocycles that satisfy $[a \cup b] = 0, [b \cup c] = 0$, and it is defined in the quotient $H(A)/aH(A) + H(A)c$. While m_3 is an operation defined for arbitrary cycles, and it depends of the homotopy h in the sense that different choices may give different representatives of the Massey product.*

So far, starting from an DGA algebra A we induced a ternary operation on V, actually by using all planar binary trees with n leaves we can create an operation m_n, for all $n > 3$, satisfying the following identity:

$$\partial m_n = \sum_{s+j+r=n} (-1)^{s+jr} m_i(id, \cdots, id, m_j, id, \cdots, id). \tag{2}$$

where in the summand $(-1)^{s+jr} m_i(id, \cdots, id, m_j, id, \cdots, id)$ the term m_j is located in the position $s+1$ and $i = s+1+r$.

Definition 1.3. *An A_∞-algebra is a complex with n-ary operations $m_n : A^{\otimes n} \to A$ of degree $n-2$ that satisfy equation (2) for all $n > 1$.*

The following example comes from [2]. Lets consider an associative algebra A, and the Hochschild cochains of A

$$C^*(A) = Hom(A^{\otimes *}, A),$$

where the differential is given by

$$\begin{aligned}\partial f(a_1, \cdots, a_{n+1}) &= a_1 f(a_2, \cdots, a_{n+1}) + (-1)^{n+1} f(a_1, \cdots, a_n) a_{n+1} + \\ &\quad + \sum (-1)^i f(a_1, \cdots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \cdots, a_{n+1}),\end{aligned}$$

Then a 3-cocycle g satisfies:

$$0 = a_1 g(a_2, a_3, a_4) - g(a_1 a_2, a_3, a_4) + g(a_1, a_2 a_3, a_4) - g(a_1, a_2, a_3 a_4) + \\ + g(a_1, a_2, a_3) a_4.$$

If we consider $A[\epsilon]/(\epsilon)^2$ with degree $\epsilon = 1$, and the product $\cdot_\epsilon$ on $A[\epsilon]/(\epsilon)^2$ induced by the product on A. Then $m_2 = \cdot_\epsilon, m_3 = \epsilon g$ induces an A_∞-structure on A, because A is associative and the cocycle condition of g is equivalent to (2) with $n = 4$.

If instead we consider a variable ϵ_n of degree $n-2$, and we take $g \in C^n(A)$ a Hochschild cocyle, then $(A[\epsilon_n]/(\epsilon_n)^2, m_2 = \cdot_\epsilon, m_n = \epsilon g)$ is an A_∞-algebra.

We saw before that when A is a DGA algebra, $H^*(A)$ inherits the structure of A_∞-algebra. The maps $H(A)^{\otimes n} \to H(A)$ are called A_∞-Massey products in [3].

In fact if we start with an A_∞-algebra structure on A then we will induce an A_∞-structure on V.

Definition 1.4. *An A^∞-morphism between A_∞-algebras (A, m_i) and (B, m'_i) is a sequence of degree $n-1$ maps f_n satisfying: $\partial(f_n) =$*

$$= \sum (-1)^{p+qr} f_k \circ (id, \cdots, id, m_q, id, \cdots, id) - \sum (-1)^{\epsilon} m_k \circ (f_{i_1}, f_{i_2}, \cdots, f_{i_k}),$$

Where f_1 is a map of complexes, in the summands $f_k \circ (id, \cdots, id, m_q, id, \cdots, id)$ m_q is located in the $p+1$ position and $i = p+1+r$, $\epsilon = (k-1)(i_1-1) + (k-2)(i_2-1) + \cdots + (1)(i_k-1)$.

So when all higher maps vanish we get $f_1(d_A) = d_B f_1$ and $f_1(m_2) = m'_2(f_1, f_1)$. Then a map of DGA algebras is in particular a map of A^∞-algebras.

Theorem 1.5. *Let (A, d_A) be an A_∞-algebra and a complex (V, d_V) that is a homotopy retract of A :*

$$h \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} V$$

satisfying $Id - ip = dh + hd$, with i a quasi-isomorphism. Then V inherits an A_∞ structure and i extends to an A_∞ map that is a quasi isomorphism.

1.0.2. *Formality.* A Differential Graded Associative algebras is formal if there is a zig zag of quasi isomorphims of DGA algebras: $(A, d) \leftarrow \cdots \to (H(A), 0)$.

Theorem 1.6. *If for a choice of the retraction*

$$h \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} H^*(A)$$

with i a quasi isomorphim, the higher A_∞-Massey products vanish, then the algebra is formal.

Proof:We saw that any homotopy retraction will induce a quasi isomorphism as A_∞-algebras, if there is h so that the A_∞ products vanish for $n > 2$, then the map is actually a map of DGA algebras.

On the other hand if a minimal DGA algebra is formal, then the higher Massey products $(n \geq 3)$ vanish (see [4]).

In the proof of theorem 1.6 the homotopy h, determines all higher Massey products, and to request that for that choice of the homotopy all higher Massey products vanishes can be paraphrased as "one may make uniform choices so that the form representing all Massey products and higher Massey products are exact". This is stronger than asking each individual Massey products to vanish.

2. ADDENDUM

I want to add a recent application of the Massey product to show that something is **not** formal. All the words that appear here will eventually be defined in the seminar.

Let O be a topological operad. We consider $C_*(O)$ singular chains complex on O and $H_*(O)$ its homology[3]. If there are cycles a, b, c so that $a \circ_i b = dx$, $b \circ_j c = dy$, then

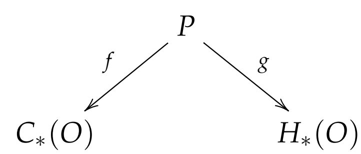
$$< a, b, c > := x \circ_{i+j-1} c - (-1)^{|a|} a \circ_i y,$$

satisfies $d < a, b, c >= 0$. We compose in the $i + j - 1$ entry to make sure this is a cycle.

Its class $[< a, b, c >]$ in cohomology is called the Massey Operadic product (of type I). Professor Muriel Livernet proved in 2014 [5] the following theorem:

Theorem 2.1. *If there exist cycles a, b, c in $C_*(O)$ such that $< a, b, c >\neq 0$ and if $H_{|a|+|b|+1}(O) = H_{|b|+|c|+1}(O) = 0$, then O is not formal.*

Sketch: Here the idea is that if we have a roof



we lift the Massey product to P and then show that g cannot be a quasi isomorphism for degree reasons.

We lift those cycles a, b, c to $a^p, b^p, c^p \in P_*$. It may happen that $f(a^p) = a_1$, when $a_1 - a = ds$, but we can show that $[< a_1, b_1, c_1 >]$ does not vanish neither. Because f is quasi isomorphism, in P we will also have $a^p \circ_i b^p = dx^p, b^p \circ_j c^p = dy^p$ and $[< a^p, b^p, c^p >]$ is non vanishing because its image $[< a_1, b_1, c_1 >]$ is non vanishing. Then $g([< a^p, b^p, c^p >])$ won't vanish. On the other hand $g(x) \in H_{|a|+|b|+1}(O) = 0$, and it is the same with $g(y)$, it follows that $g([< a^p, b^p, c^p >]) = 0$ which is a contradiction.

From here follows non formality of Swiss Cheese operad for $d > 3$ by finding explicit Massey operadic products. How to find those explicit Massey products? maybe it may help to talk about the case $d = 2$ which uses the so called "Eye Law".

Consider the upper half plane H on C together with the real line. And consider the space of different points $a, b \in H$ module the action of the group $x \to rx + s, r, s \in R, r > 0$. It will be clear in the future that there is a compactification of this space (lets call it the "Eye"), where the boundary is given by allowing those two points to collide or to go to infinity.

Lets assume that a is mapped to i, then if b approaches a, different angles will give different points in the boundary of the Eye (and form the pupil).

[3]From now on we are following definitions and notations of [5].

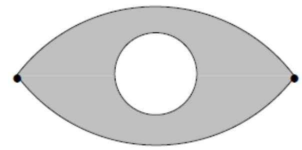


FIGURE 4. Compactification of the space of 2 points in the upper plane. See [6].

The other boundary below is given by letting b go to the real line (forming the lower eyelid), or to ∞, which we prefer to think as sending b to i and letting a go to the real line (forming the upper eyelid).

To describe the Eye Law assume we move around the pupil, then the two points turn around each other in H. If instead we move around the eyelids, then a is in H while b in R, moving from $-\infty$ to ∞, eventually a falls to the real line with $a < b$ and then b goes to H, while a moves from ∞ to $-\infty$, then b falls to the real line with $b < a$, and we return to the original position.

Assuming a map from the operad of Swiss Cheese to its cohomology, in [5] they found chains $f \circ l$ and $< a; f, f >_{II}$, with $f \circ l$ resembling the two points rotating in H around each other, and $< a; f, f >_{II}$ describing the external boundary of the "eye" where $<, >_{II}$ is a variant of the operadic Massey product. Because of the pupil and the eyelids are boundary of a 2-cell, there is a homotopy m so that $\partial m = \pm < a; f, f > \pm f \circ l$. We could see that $f \circ l$ is non trivial, and so the product $< a; f, f >_{II}$ does not vanish. But for the same argument about the degree that we used before, the image of $< a; f, f >_{II}$ in homology of the Swiss Cheese goes to zero, giving a contradiction.

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