Genus-zero TQFTs and operads.

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0.1 Forgetting structure

In a topological quantum field theory we have a functor F that assigns to S^1 a vector space $F(S^1)$. The cobordism of the pair of pants define a (commutative) product $F(S^1)^{\otimes 2} \to F(S^1)$.

In this talk I want to describe some operads that help us understand the algebraic structure that $F(S^1)$ inherits. We will restrict to cobordisms from $\cup_n S^1$ into S^1 of genus 0. This is necessary to avoid maps of the form $V \to V^{\otimes m}$. A general cobordism can be described as a composition of cobordisms of the form $\cup_n S^1 \to S^1$ with a cobordism $S^1 \to S^1$ possible of higher genus.

A TQFT will assign to the interval I operations $F(I)^{\otimes n} \to F(I)$ induced by cobordisms from disjoint copies of the interval to an interval, here we work with cobordisms without holes. There is no orientation preserving transformation between the cobordisms in figure 1, which implies that the product on $F(I)$ may not be commutative.

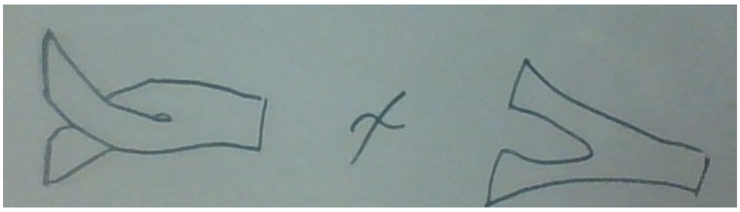

Figure 1: $F(I)$ may not be commutative.

In the previous paragraphs we have some vector space V together with elements of $Hom(V^{\otimes n}, V)$ that satisfy rules of composition. An operad O is a sequence of vector spaces $O(n)$ (or topological spaces or chain complexes, etc), together with compositions that satisfy some conditions that I won't explain. It is enough to say that $End_v(n) = Hom(V^{\otimes n}, V)$ is our model of how does an operad behave.

0.2 Examples of operads

We are considering only cobordisms from $\cup_n S^1 \to S^1$, from now on instead of drawing cobordisms we imagine that we look through the waist. What we will see is n disk inside of the waist. There is an operad called the little disk operad such that for every n, $E_2(n)$ is the configuration space of n disk into the unit disk. Composition is given by plug in d-disks into the unit d-disk. Notice that this is compatible with what we will see if we do the composition of cobordisms. The same arguments as before explain why E_2 parametrize commutative operations (in some sense).

Open-close cobordism. Lets consider the cobordism from $\cup_3 S^1 \to S^1$ that is symmetric with respect to the z-coordinate. Because of symmetry we can just cut the cobordism along the horizontal line. If we situate our eyes on the waist of the cobordism and we cut the cobordism what we will see now resembles a Swiss Cheese.

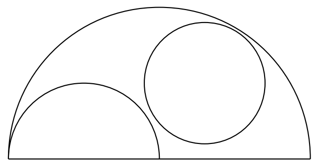

Figure 2: Open closed cobordism seen from the wrist.

The Swiss Cheese operad[1] consist of the space of embeddings of disjoint disks and half disks into the unit half disk. We think of the cobordism from S^1 to I as an action of $F(S^1)$ on $F(I)$. We deduce that we have a map $h : F(S^1) \to ZF(I)$ where $ZF(I)$ is the center of the algebra $F(I)$. [2]

Allow me to move to the algebraic side by considering singular chains to obtain operads on chain complexes. Composition operations of topological operads induce composition on operad of chain complexes:

$$c_{-*}O(n) \otimes c_{-*}O(m) \to c_{-*}(O(n) \times O(m)) \to c_{-*}O(n+m-1).$$

The first map is the Eilenberg Zilber map, which says that if you have a simplicial structure on an space A and another on B then there is way to induce a simplicial structure on $A \times B$.[3]

The topological world is very convenient in terms of intuition. What can the algebraic side show to us?

0.3 Algebras

Lets consider an algebra over *homology* of the previous operads.

$H_*(E_2)$ is generated by $\mu \in H_0(E_2)$ and $\{\} \in H_1(E_2)$. We can give a visual proof that the bracket satisfies

$$\{ab, c\} = a\{b, c\} + \{a, c\}b,$$

this is called Leibnitz relation, where $ab := \mu(a, b)$. Gerstenhaber proved that for an associative algebra A, Hochschild cohomology $H^*(A)$ has a commutative product (the cup product), and a Lie braket which satisfy Leibnitz relation.

[1] Strictly speaking we are dealing with a two colored operad.
[2] We forget the map from the interval to S^1.
[3] The EZ map has an inverse called the Alexander Withney map, both maps preserve the monoidal structure but only the EZ map is symmetric.

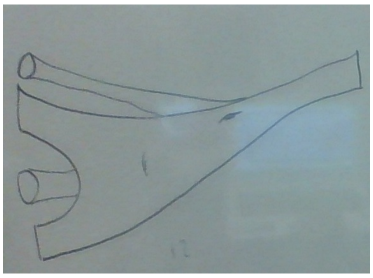

Figure 3: Here inputs are displayed on the left side and the output to on right side.

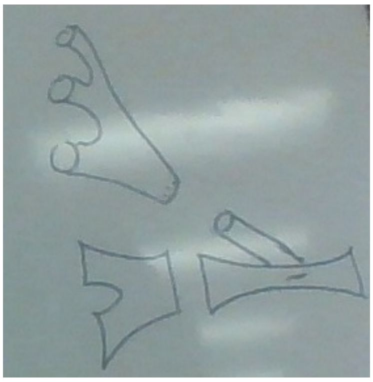

Figure 4: Composition of open-close cobordisms.

What about algebras over the homology of the Swiss Cheese operad? They are pairs (B, A) where $H_*(E_2)$ acts on B, H_*E_1 acts on A and there is a central morphisms $h : B \to A$. For the case of $(H^*(A), A)$ the central morphisms $h(f)$ is multiplication by $Z(A)$ if $f \in H^0(A)$ or zero otherwise.

Another operad that I want to introduce is the operad of Cylinders[4], we think of cylinders $S^1 \times [a, b]$ with market points $I \in S^1 \times a, O \in S^1 \times b$. We can decorate the sides of the cylinder with little disk, and we can also combine two cylinders into a big cylinder, we just rotate the upper cylinder so that the O point of it coincides with the I point of the cylinder below. We denote by $Cyl(n, 0) = E_2(n)$, by $Cyl(0, 1)$ the space of configuration of cylinders with market points I, O, and by $Cyl(n, 1)$ the space of configuration of n disk on the surface of a labeled disk.

We have the usual operations $\mu, \{,\}$ coming from $H_*(E_2)$, an operation i on $H_0(cyl(1, 1))$ that inserts a disk on the surface of the disk. $l \in H_1(cyl(1, 1))$ which represents the cycle that rotates a disk on the cylinder along the vertical axis. There is also an operation $\delta \in H_1(Cyl(0, 1))$ that rotates the base of the cylinder.

For example l can be archived by rotating both the disk and the base point O to the right (δi) and then rotating the base point to the left $(-i\delta)$, we conclude that $l = \delta i - i\delta$. l is called the Lie derivative. Similarly we can verify that:

- $i_{[a,b]} = [i_a, l_b]$,

[4]This is a two colored operad.

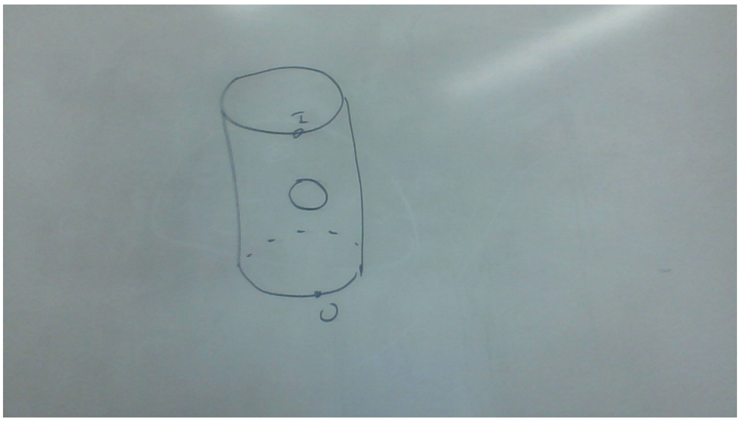

Figure 5: An example of a term in $c_0Cyl(1,1)$.

- $l_{\mu(a,b)} = l_a i_b + i_a l_b$,
- $\delta^2 = 0$.

For more details see [DTT08].

Algebras over H_*Cyl are pairs (V, W) were V is a Gerstenhaber algebra with the operations $\mu, \{,\}$, we have $i : V \otimes W \to W, l : V[-1] \otimes W$, and $\delta : W \to W$ satisfying the previous conditions. $(H^*(A), H_*(A))$ is an example of an algebra over the homology of the cylinder operad, probably $(\oplus \wedge^n Vect(X), \Omega(X))$ is a more well known example.

0.4 Deligne conjecture.

Deligne asked if the action of $H_*(E_2)$ on $H^*(A)$ comes from an action at the level of chains? we wonder if we can find an action of c_*E_2 on $Hoch^*(A)$ that reduces to the previous one. Kontsevich asked the same question for the SC and Cyl operads.

Warning: [Liv14] The Swiss Cheese[5] operad is not formal. More or less the reason is that the operation given by rotating two disk inside of the half disk can be realized as a composition in two different ways, this lead to an algebraic relation that cannot happen on the H_*SC_2.

Theorem 0.4.1 *[DTT06] Let A be an associative algebra. The natural operad on the pair $(Hoch(A), A)$ is quasi isomorphic to c_*SC_2, the induced action on homology recovers the structure previously defined.*

Theorem 0.4.2 *[KS06] Let A be an unital associative algebra. The pair $(Hoch^*(A), Hoch_*(A))$ is an algebra over an operad quasi isomorphic to c_*Cyl.*

There are higher analogs of the E_d, SC_d operads, and given M a framed $n-1$-manifold, [Hor13] defined the notion of the operad Cyl_M cylinder over M. The topological higher analogs of the previous theorems are:

[5] There are three versions of the Swiss Cheese operad, this statement is about Kontsevich's version.

Theorem 0.4.3 *[Tho10] Let A be an algebra over the topological E_{d-1} operad. The pair $(Hoch^*(A), A)$ is weak equivalent to an algebra over the topological SC_d, and it is initial in the category of such pairs (B, A).*

Theorem 0.4.4 *[Hor13] Given A algebra over the topological operad E_d, M a framed $d-1$ manifold, then $(Hoch^*(A), \int_M A)$ is weak equivalent to an algebra over Cyl_M.*

In my thesis I proved that:

Theorem 0.4.5 *Let A be an algebra over c_*E_{d-1}. The pair $(Hoch^*(A), A)$ is weak equivalent to an algebra over the c_*SC_d, and the pair is initial in the category of such pairs.*

This combined with results from [Hor13] implies that for M a framed $d-1$-manifold, $(Hoch(A), \int_M A)$ is an algebra over c_*Cyl_M.

0.5 Remembering structure

Let A be an associative algebra. If we define $F(I) = A$, then it makes sense to define $F(S^1) := Hoch^*(A)$. When can we extend this to a tqft? At least we need A "Frobenius" algebra with $Hoch^*(A)$ "Frobenius" (which corresponds to those cobordisms that we ignored at the very beginning).

References

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