Braid groups actions on Categories.

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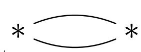
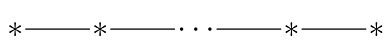
This are my notes of Licata's course 'Braid group actions on categories', for the seminar:Associators, Formality and Invariants at NU.

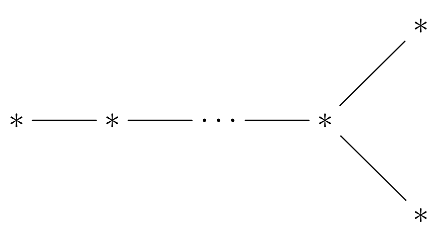
There are typos and probably some statements/proofs are not correct, I take responsability for that. I appreciate the help of 温欣, from ECNU, Shanghai.

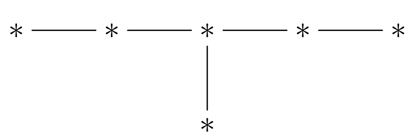
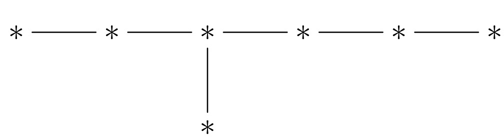
0.1 Semisimple lie algebras.

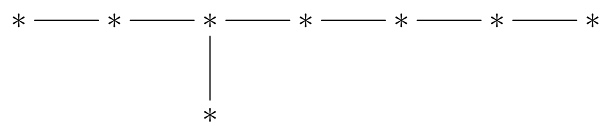
Let Γ be a finite graph without multiple edges  and no loops, we denote by I its vertex set, and by E its edge set.

In particular we will consider simply laced Dynkin diagrams:

- A_n 
- D_n



- E_6 
- E_7 

- E_8 

To Γ we associate the Weyl group:

$$W_\gamma =< s_i | s_i^2 = 1, \begin{array}{c} s_i s_j = s_j s_i, \quad i,j \notin E, \\ s_i s_j s_i = s_j s_i s_j, \quad i,j \in E \end{array} >_{i \in I}$$

For example, when $\Gamma = A_1 = *$, then

$$W_\Gamma =< s | s^2 = 1 >\sim Z/2Z.$$

and when $\Gamma = A_{n-1}$,

$$W_\Gamma \sim S_n,$$

$$s_i \to (i \;\; i+1).$$

Question. For which Γ is $|W_\Gamma| < \infty$?[1]

Answer: Theorem [Coxeter] W_Γ is finite iff every connected component of Γ is of the kind A, D or E.

Let B_Γ the Artin Tier Braid group defined by the braid relations

$$< \sigma_i | \begin{array}{c} \sigma_i \sigma_j = \sigma_j \sigma_i, \quad i,j \notin E, \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad i,j \in E \end{array} >_{i \in I} .$$

Lets denote

$$\pi : B_\Gamma \to W_\Gamma.$$

Let B_p^+ be the monoid presented by $\{\sigma_i |$ braid relations $\}$, we also have a morphism of monoids

$$\kappa : B_p^+ \to B_p.$$

It is not obvious that κ is injective, as it turns out to be[2], and we can consider the so called positive braids $\kappa(B_p^+) = B_p^+ \subset B_p$. We have the following diagram:

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[1]It is easy to see that it suffices to consider Γ connected, as $\Gamma \simeq \Gamma_1 \cup \Gamma_2$ implies $W_\Gamma \simeq W_{\Gamma_1} \times W_{\Gamma_2}$.

[2]Proven by Deligne in ADE case, and by Palis in general (2005) .

$$\begin{array}{ccc} \pi : B_\Gamma & \twoheadrightarrow & W_\Gamma \\ \uparrow & \overset{\rho}{\swarrow} & \\ B_\Gamma^+ & & \end{array}$$

Where $\rho : W_\Gamma \to B_\Gamma^+$ is a map of sets defined as follows:

Given $w \in W$ write it $w = s_{i_1} \cdots s_{i_k}, k$ small as possible, that is $k = l(w)$ minimal lenght, then $\rho(w) = \sigma_{i_1} \cdots \sigma_{i_k}$.

0.1.1 Note:

$$\begin{aligned} \rho(1) = 1, &\qquad l(1) = 0; \\ \rho(s_i) = \sigma_i, &\qquad l(s_i) = 1; \\ 1 = s_i s_i &\subset W, \\ 1 = \rho(1) \quad \neq &\quad \rho(s_1 s_1) = \sigma_i^2 \subset B_\Gamma \end{aligned}$$

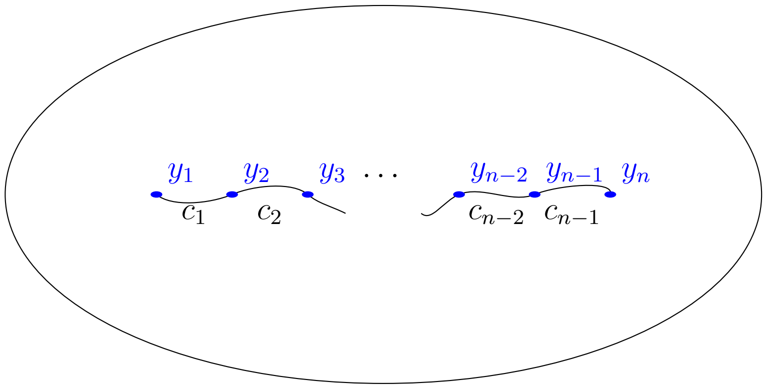
0.1.2 Examples:

- If $\Gamma = A_1, W_{A_1} \sim Z/2Z, B_{A_1} \sim Z$.
- What distinguishes B_Γ when $\Gamma = A, D, E$ form other cases? Conjecture: $Z(B_\Gamma) \neq 0$ iff $\Gamma =$ type A, D, E. [3]
- If $\Gamma = A_{n-1}, B_{A_{n-1}}$ is topological in nature, it occurs as a mapping class group:

  Lets consider the disk with n market points $(D, \{y_1, \cdots .y_n\})$.

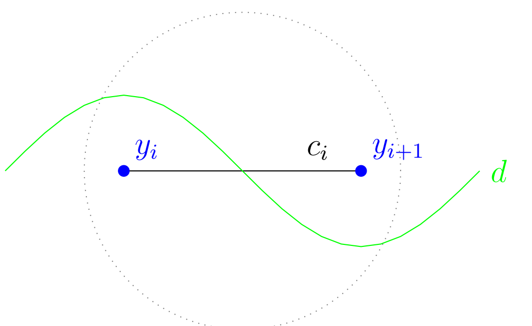
  Lets fix "nice" paths $c_i : y_i \to y_{i+1}$.

[3] It is known that $Z(B_\Gamma) = Z, \Gamma = A, D, E$. In fact there is an element $w_0 \in W_\Gamma, \rho(w_0) = \triangle \in B_\Gamma^+, Z(B_\Gamma) = \triangle^2$.

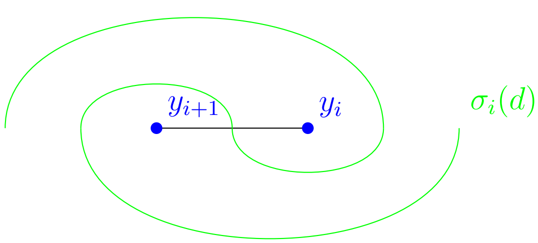


$$
\begin{aligned}
B_{A_{n-1}} &\sim MCG(D, \{y_1, \cdots .y_n\}, \partial D) \\
&= \{ \text{ homomorphisms } : D \to D, \text{ which are id on } \partial D \\
&\quad \text{ and preserve the set} \{y_1, \cdots, y_n\}\}/\text{isotopy.}
\end{aligned}
$$

The isomorphism is given by: σ_i is sent to half Denh twist of c_i clockwise, notice that this permutes the set $\{y_i\}$. so given a curve d:



then we obtain:



Alternatively,

$B_{A_{n-1}} = \pi_1(P_n; \{y_1, \cdots, y_n\})$, where

$$P_n = D^{\times n} - \cup_{i \neq j}\{(x_1, \cdots, x_n) | x_i = x_j\}/S_n.$$

0.2 A representation of W_Γ

Let $V_Z = span_Z\{p_i\}_{i \in I}, V_Q = V_Z \times Q = span_Q\{p_i\}_{i \in I}$. We'll define a map of W_Γ on V_Q (and V_Z) as follows: For $i \in I$, define $q_i : V_Z \to Z$ by

$$q_i(p_j) = \begin{cases} -2 \text{ if } i = j, \\ 1 \text{ if } i, j \in E, \\ 0 \text{ if } i, j \notin E. \end{cases}$$

(Cartan Matrix).

We define a map $\rho : W_\Gamma \to End(V)$ by $\rho(s_i) = 1 + p_i q_i$

0.2.1 Exercise

Check that $\rho(s_i)\rho(s_i) = 1$ (as expected since $s_i^2 = 1$) and that ρ defines a representation of W_Γ. Compare with Note(0.1.1).

Theorem.The Representation $\rho : W_\Gamma \to GL(V)$ is faithful (ρ is injective). How to construct a representation of B_Γ?

Let $V_{Z[t,t^{-1}]} = Span_{Z[t,t^{-1}]}(p_i)_{i \in I}, V_{Q[t]} = V_{Z[t,t^{-1}]} \otimes_{Z[t,t^{-1}]} Q(t)$.

Define $q_i : V_{Z[t,t^{-1}]} \to Z[t, t^{-1}]$ by

$$q_i(p_j) = \begin{cases} t + t^{-1} \text{ if } (i = j) \\ 1 \text{ if } i, j \in E, \\ 0 \text{ if } i, j \notin E. \end{cases}$$

similarly, define

$\rho(\sigma_i) : V_t \to V_t$ by

$$\rho(\sigma_i) = 1 - t p_i q_i, \rho(\sigma_i^{-1}) = 1 - t^{-1} p_i q_i.$$

t deformation from V_Z to $V_{Z[t,t^{-1}]}$.

It follows that $\rho(\sigma_i)\rho(\sigma_i^{-1}) = 1$.

This is the Reduced Brauer representation of B_Γ, sadly, the reduced Brauer representation is almost never faithful: For A_1, A_2 it is faithful, for $A_n, n > 3$ it is not faithful, and it is not know for A_3.

length	element
3	$s_1 s_2 s_1 = s_2 s_1 s_2,$
2	$s_1 s_2, s_2 s_1,$
1	$s_1, s_2,$
0	1.

Table 1: Bruhat lenght on W_Γ.

How can one find a better situation to study B_Γ? linear structure?

Lets try to find a faithful representation of B_Γ in some other vector spaces, Bigelew, Krammer and others provide such a vector space, which is infinite dimensional unless Γ is ADE.

We will try to find a representation of B_Γ in a linear category related to Brauer representation.

Problem: Compute the center of B_Γ, is B_Γ torsion free? (known in type ADE)

Brauer representation of $B_{A_{n-1}}$ is closely related to the appear of $B_{A_{n-1}}$ as a $MCG(X_n)$.

Clarify this!

Up shot: Brauer representation is related to the topology of X_n.

Let's calculate

$$\begin{aligned}
\Gamma &= A_2, \\
V &= spam\{p_1, p_2\}, \\
W_{A_2} &= \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}, \\
&= S_3, \\
\psi(s_1)(p_1) &= -p_1, \\
\psi(s_1)(p_2) &= p_2 + p_1.
\end{aligned}$$

We need the Bruhat lengh on W_Γ :

It has the following proprety: given $w \in W_\Gamma$, if $l(w) = k$ then $l(ws_1) = l(w) + 1$ or $l(ws_1) = l(w) - 1$.

Proposition: For all $w \in W_\Gamma, i \in I, l(ws_i) > l(w)$ implies $w\dot{p}_i > 0$. $l(ws_i) < l(w)$ implies $w\dot{p}_i < 0$.

Proof: Exercise.

Now let $p = \sum p_i \in U$.

Corollary (Bjorner-Bruti):

$$\{s_i | c_i < 0 \text{ with } wp = \sum c_i p_i\} = \{s_i | l(ws_i) < l(w)\} := D(w) \subset \{s_i\}_{i \in I}$$

If $u \neq e$ then $up \neq p$ (as $D(w) = \emptyset$.)

1 Second Day.

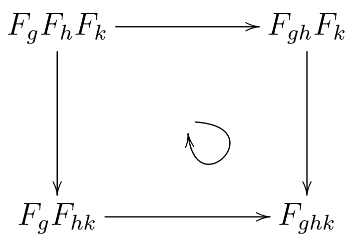
Categorical Braid Group Actions.

Let G be a group and C a category, by a weak action of G on C we mean:

$g \in G \longrightarrow F_g : C \to C, \qquad$ for $g, h \in G: \; F_f F_h \simeq F_{fh} \qquad F_1 = Id.$

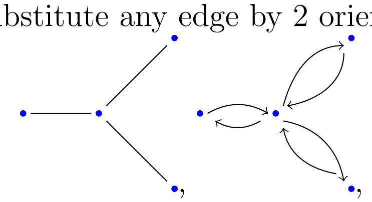
where $\simeq$ means that there is a natural transformation.

A genuine action require a little bit more, namely the commutativity up to a natural transformation of:



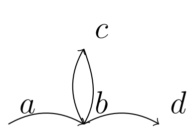
1.1 Zig Zag Algebra A_Γ

Let Γ be a finite graph, and consider $\bar{\Gamma}$ the double quiver of Γ. That is substitute any edge by 2 oriented edges in opposite directions.



Let Path($\bar{\Gamma}$) denote the Path algebra of $\bar{\Gamma}$, $span_C\{$ finite lenght paths in $\bar{\Gamma}\}$. A path x is completely determined by specifying the vertex you travel to, along x.

1.1.1 Example

$(a|b|c|b|d)$ describes 

, while the constant path is $(d) = e_d$, notice that $e_d e_d = e_d$,[4] and that if there is at least one edge, then the algebra has

[4]Multiplication in Path is concatenation of paths: $(a|b|c)(x|y) = (a|b|c|y)$ if $x = c$ or $(a|b|c)(x|y) = 0$ otherwise.

infinite elements.

Path lenght induces a grading on Path($\bar{\Gamma}$).

Definition:

- For $\Gamma = *$, let $A_\Gamma = C[x]/x^2,\ \deg x = 2$.
- For $\Gamma = * - *$, let $A_\Gamma = Path(\bar{\Gamma})/\{$ all leght 3 paths$\}$.
- For any other Γ, let A_Γ be a quotient of Path($\bar{\Gamma}$) by the 2 sided ideal generated by:

$$(a|b|c) = 0 \text{ if } a \neq c,$$
$$(a|b|a) = (a|c|a) \text{ when } b \neq c \text{ are both connected to } a.$$

A_Γ[5] is a finite dimentional graded C algebra.

Facts about A_Γ :

- A_Γ is a symmetrical algebra.
- A_Γ is Koszul iff Γ is not a finite type ADE.

1.1.2 Exercise: compute its graded dimension.

Let e_i be the idempotent (lenght 0 path at $i \in I$), $e_i e_j = \delta e_i \in A_\Gamma$.

Set $P_i = A_\Gamma e_i = span_C\{$ path which ended at $i\}$, it is a graded left $A_\Gamma - mod$ and $Q_i = e_i A_\Gamma = span_C\{$ path which starts at $i\}$, it is a graded right $A_\Gamma - mod$. From now on $Vect$ denotes the category of graded C vector spaces.

One can associate to P_i a functor

$$\begin{aligned} F_{P_i} : Vect &\rightarrow A - \Gamma Mods \\ V &\mapsto P_i \otimes_C V \\ F_{Q_i} : A - \Gamma Mods &\rightarrow Vect \\ M &\mapsto Q_i \otimes_C M \end{aligned}$$

From now on we are going to switch the internal degree of A_γ by 1, in such a way that the idempotent e_i has degree -1, $e_i e_{i+1}$ has degree 0 etc.

[5] this skew version give us a formal neighborhod

Lets compute

$$Q_iP_j := Q_i \otimes_{A_\Gamma} P_j = \begin{cases} C<-1> \oplus C<1> & i=j \\ C & i,j \in E \\ 0 & i,j \notin E \end{cases}$$

We denote by $Com(A_\Gamma - mod)$ the homotopy category of complexes of A_Γ-modules. Lets remember that $Kom(A)$ has as Objects complex of A_Γ mod, $\cdot M_i \xrightarrow{\partial_i} M_{i+1} \xrightarrow{\partial_{i+1}} \cdot$ where the boundary maps are homogeneous of degree 0. This is an abelian category. Lets consider Null hom:= the set of hull homotopic chain maps, and we define $Com(A_\Gamma - mod) = Kom(A - \Gamma - mod)/(nullhom)$.

Example: $x := 0 \longrightarrow C \xrightarrow{1} C \longrightarrow 0 \longrightarrow \cdot$

$y := 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdot$

$x \neq y \in Kom, x \simeq y \in Com.$

Goal:we are going to construct an action in $Com(A_\Gamma)$ of the braid group, we need:

$$\Phi_i, \Phi_i^{-1} | \begin{array}{rcl} \Phi_i\Phi_j = & \Phi_j\Phi_i & i,j \notin E \\ \Phi_i\Phi_j\Phi_i = & \Phi_j\Phi_i\Phi_j & i,j \in E \\ \Phi_i^{-1}\Phi_i \simeq & id & \\ \simeq & \Phi_i\Phi_i^{-1}, & \end{array}$$

since we are working with groups we also need Φ_i to be equivalence of categories.

How to proceed? to get functors $F : Com(A_\Gamma) \to Com(A_\Gamma)$ we consider complexes of $(A_\Gamma, A_\Gamma)-$bimodules.

Goal: for each $i \in I$ to define a complex of $(A_\Gamma, A_\Gamma)-$bimodules B_i^* and from them obtain a functor:

$$\Phi_i = B_i^* \otimes \_ : Com(A_\Gamma) \to Com(A_\Gamma).$$

1.1.3 Examples of $(A_\Gamma, A_\Gamma)-$ bimodules.

A_Γ itself, the corresponding fucntor $A_\Gamma \otimes_{A_\Gamma} \_$ is naturally equivalent to the Id, so we will refer to it as Id from now on.

$P_i \otimes_C Q_j$ is another $A_\Gamma - A_\Gamma$ bimod while $Q_j \otimes_C P_i \in (C, C)-$ bimod.

Looking at the Brauer representation, We want to categorize $\sigma_i = 1-tp_iq_i$, and a general Yoga tell us that minus signs lead us to complexes, lets consider:

$$B_i = \quad \{0 \longrightarrow 0 \longrightarrow A_\Gamma \longrightarrow P_iQ_i < 1 > \longrightarrow 0\},$$

$$B_i' = \quad \{0 \longrightarrow P_iQ_i < -1 > \longrightarrow A_\Gamma \longrightarrow 0 \longrightarrow 0\}$$

$$\Phi_i = B_i \otimes \_, \Phi_i^{-1} = B_i' \otimes \_,$$

Notice that we will have a double grading, $A < n > [m]$ means that the elements of the algebra have degree n and as a complex, it is located at m. Also, in the case of B_i, B_i' both grading coincide.

$Hom_{(A_\Gamma, A_\Gamma)}(P_iQ_i < -1 >, A_\Gamma) \simeq C$ as by definition we have to construct a map from $A_\Gamma e_i \otimes_C e_i A_\Gamma \to A_\Gamma, x \otimes y \mapsto xy$.

By adjuntion properties $Hom_{(A_\Gamma, A_\Gamma)}(A_\Gamma, P_iQ_i < 1 >) \simeq C$ as well.

How to check that we satisfy the Braid relations?

We need to prove $\Phi \otimes_{A_\Gamma} \Phi^{-1} \simeq Id$.

1.1.4 Exercise

Let B be a (R-S)bimodule, and $F_B = B \otimes \_ : S - Mod \to R - Mod$. $f : B \to B'$ a bimod map. Then f induces a natural transformation of functors $F_f : F_B \to F_{B'}$.

It suffices to check that

$B_i \otimes_{A_\Gamma} B_i' \simeq A_\Gamma = Id = B_i' \otimes_{A_\Gamma} B_i$ in the homotopic category of (A_Γ, A_Γ) bimodules.

Similarly

$$B_i \otimes_{A_\Gamma} B_j \otimes_{A_\Gamma} B_i \simeq B_j \otimes_{A_\Gamma} B_i \otimes_{A_\Gamma} B_j, \; i, j \in E \tag{1}$$

$$B_i \otimes_{A_\Gamma} B_j \simeq B_j \otimes_{A_\Gamma} B_i, \; i, j \notin E. \tag{2}$$

which is due to Khovanov-Seidel, Rouquier Zimmermann, circa 2001.

1.2 Third day.

Theorem [Khovanov-Seidel, Rouquier Zimmermann, circa 2001.]

(1) and (2) hold, and

$B_i \otimes_{A_\Gamma} B'_i \simeq A_\Gamma \simeq B'_i \otimes_{A_\Gamma} B_i$.

This implies that the braid group B_Γ acts on $Com(A_\Gamma)-$mods.

Sketch:

We will need an auxiliar result:

Proposition (Gauss elimination)

Let $\longrightarrow U \xrightarrow{\begin{pmatrix} u_x \\ u_y \end{pmatrix}} X \oplus Y \xrightarrow{\begin{pmatrix} f & g \\ h & k \end{pmatrix}} X \oplus Y' \xrightarrow{\begin{pmatrix} u_x & u_y \end{pmatrix}} V \longrightarrow$ and suppose that $f : X \to X$ is an isomorphism, then this complex is isomorphic in the homotopy category to:

$$\longrightarrow U \longrightarrow Y \xrightarrow{k-hf^{-1}g} Y' \longrightarrow V \longrightarrow$$

Now lets make some calculations:

$$\begin{aligned}
B_i B'_i &= (P_iQ_i < -1 > \to A_\Gamma) \otimes_{A_\Gamma} (A_\Gamma \to P_iQ_i < 1 >) \\
&= P_iQ_i < -1 > \to P_i(Q_iP_i)Q_i \oplus A_\Gamma \to P_iQ_i < 1 > \\
&= P_iQ_i < -1 > \to P_i(C < 1 > \oplus C < -1 >)Q_i \oplus A_\Gamma \to P_iQ_i < 1 > \\
&= P_iQ_i < -1 > \to P_iQ_i < 1 > \oplus P_iQ_i < -1 > \oplus A_\Gamma \to P_iQ_i < 1 > \\
&= 0 \to A_\Gamma \to 0
\end{aligned}$$

Where we used the previous proposition, it is important to check that infact the corresponding maps are isomorphisms so that we can substitue the complex for an homotopical one. In a similar way it is proven that $B'_iB_i \simeq A_\Gamma$.

Now suppose $i, j \notin E$,

$$\begin{aligned}
B_iB_j &= (P_iQ_i < -1 > \to A_\Gamma) \otimes_{A_\Gamma} (P_iQ_i < -1 > \to A_\Gamma) \\
&= (P_i(Q_iP_j)Q_j < -2 > \to P_iQ_i < -1 > \oplus P_jQ_j < -1 > \to A_\Gamma) \\
&= 0 \to P_iQ_i < -1 > \oplus P_jQ_j < -1 > \to A_\Gamma) \\
&= B_jB_i
\end{aligned}$$

and finally, lets suppose $i, j \in E$:

$$B_iB_jB_i = (P_iQ_i < -1 > \to A_\Gamma) \otimes_{A_\Gamma} (P_jQ_j < -1 > \to A_\Gamma) \otimes_{A_\Gamma} (P_iQ_i < -1 > \to A_\Gamma)$$

$$
\begin{array}{ccccccc}
 & & P_i(Q_iP_i)Q_i<-2> & & P_iQ_i<-1> & & \\
 & & \oplus & & \otimes & & \\
=(P_i(Q_iP_j)(Q_jP_i)Q_i<-3> & \longrightarrow & P_i(Q_iP_j)Q_j<-2> & \longrightarrow & P_jQ_j<-1> & \longrightarrow & A_\Gamma \\
 & & \oplus & & \oplus & & \\
 & & P_j(Q_jP_i)Q_i<-2> & & P_iQ_i<-1> & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & P_iQ_i<-2> & & P_iQ_i<-1> & & \\
 & & \oplus & & \oplus & & \\
=P_iQ_i<-3> & \longrightarrow & P_iQ_j<-2> & \longrightarrow & P_jQ_j<-1> & \longrightarrow & A_\Gamma \\
 & & \oplus & & \oplus & & \\
 & & P_jQ_i<-2> & & P_iQ_i<-1> & & \\
 & & \oplus & & & & \\
 & & P_iQ_i<-1> & & & &
\end{array}
$$

$$
\begin{array}{ccccccc}
=0 & \longrightarrow & P_iQ_j<-2> & \longrightarrow & P_jQ_j<-1> & \longrightarrow & A_\Gamma \\
 & & \otimes & & \oplus & & \\
 & & P_jQ_i<-2> & & P_iQ_i<-1> & &
\end{array}
$$

$$
=B_jB_iB_j.
$$

1.2.1 Example

:

Let $\Gamma = * - * - * = A_3$, so we have $P_1, P_2, P_3 \in Com(A_\Gamma - Mod)$, and we got the table:

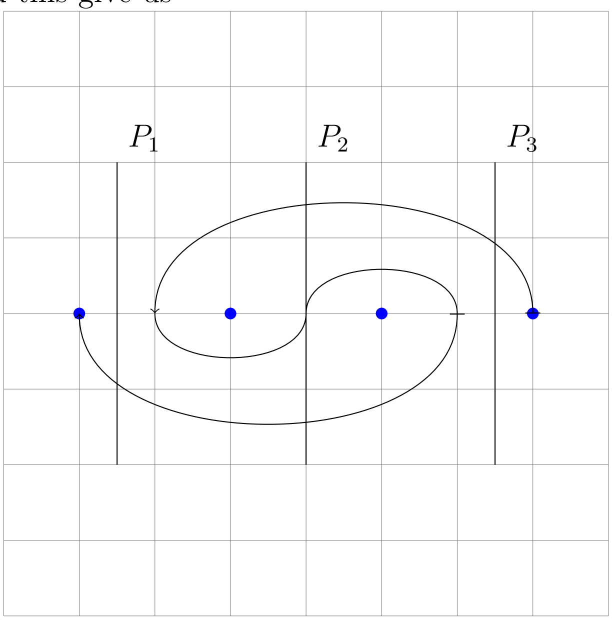
$\otimes$	P_1	P_2	P_3
Φ_1	$P_1<-2>[-1]$	$P_1<-1>\to P_2$	P_3
$\Phi_2\Phi_1$	$P_2<-3>\to P_1<-2>$	$P_1<-1>[-1]$	$P_2<-1>\to P_3$
$\Phi_1\Phi_2\Phi_1$	$P_2<-3>[-1]$	$P_1<-3>[-2]$	$P_1<-2>\to P_2<-1>\to P_3$

Theorem: (Khovanov-Seidel) If Γ is type A then the braid group action is faithful, that is $\Phi_B \simeq Id$ iff $B \simeq 1 \in B_{A_n}$.

To prove it, lets rememeber the action of B_A on Mapping Class Group see (0.1.2).

1.2.2 Example:

Let c be a curve in D with end points on the market points. Morally, to c we assign $P(c) \in Com(A_\Gamma - Mod)$ as follows: We give an orientation to c lines counter clockwise. And we assign projective mods to the intersection with the vertical lines, under some rules an internal degree shift is assigned, and this give us



$$= 0 \longrightarrow P_1 < -1 > \longrightarrow P_2 < 0 > \longrightarrow P_2 < 2 >$$

$$\otimes \qquad \oplus$$

$$P_3 < -1 > \qquad P_2 < 0 >$$

we obtain a map from

$$\{ \text{ curves } D \text{ with end points on marked points}\} \rightarrow \{ \text{ complexes of } A_\Gamma - Mod\}$$

and notice that B_Γ acts in both sets, the theorem is that this assignment interwines the 2 actions. As a corollary, the KSRZ action is faithful in type A_n.

Proof:(Sketch) Let $\beta \in B_\Gamma$ such that β acts as Id in KSRZ, so $\beta(P_i) \simeq P_i \forall i$. then (Purely topological argument):

$\beta(c_i) = c_i$, in MCG. So we are looking for brands β that take all c_i to itself, it turns out that β commutes with Dehn twist, but since they are generators this is equivalent to say that β is central, so it is a power of Δ^{2k}, but this acts by shifts so $\beta = \delta^0 = 1$.

Note: the decategorified action, is not faithful, that is, if we pass to Grothendieck group, then we get the Burau representation which is not faithful.

1.2.3 Conjecture:

The action of B_Γ is faithful for all Γ. So far we know that the action is faithful for γ ADE (Brav-Thomas).

Let B_Γ^+ denote the positive braid monoid:

$$B_\Gamma =< \sigma_i | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } i,j \notin E, \sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i \text{ if } i,j \in E > .$$

There is (an injective) monoid morphism:

$$\begin{array}{cc} \sigma_i & \in B^+ \\ \downarrow \uparrow e & \downarrow \\ s_i & \in W_\Gamma \end{array}$$

where e is a morphism of sets defined as follows: given $w \in W$, write it in reduced expression as a product of generators $w = s_{i_1} \cdots s_{i_k}, l(w) = k$ is sent to $e(w) = \sigma_{i_1} \cdots \sigma_{i_k}$.

An important construction in Braid theory is the Garside normal form of a positive braid $\beta \in B_\Gamma^+$:

We say that β' is a right factor of β if $\beta = \beta''\beta'$ with $l(\beta) = l(\beta'') + l(\beta')$, $l(\beta) = min_k\{\beta = \sigma_{i_1} \cdots \sigma_{i_k}\}$.

Garside proved that any positive braid $\beta \in B^+$ has a unique longest right factor of the form $e(w), w \in W$. So once we have $e(w_1)$ the unique longuest right factor of β in $Im(e)$ we continue inductively to get the Garside normal form $\beta = e(w_k)e(w_{k-1}) \cdots e(w_1)$, in this case we define the Garside lenght of β as $Gl(\beta) = k$.

Remark: this is defined only for positive braids $\beta \in B_\Gamma^+$.

1.2.4 Examples:

- $Gl(\sigma_1\sigma_2\sigma_1) = 1$ as $\sigma_1\sigma_2\sigma_1 \in e(s_1s_2s_1)$.
- $Gl(\sigma_1\sigma_2\sigma_2) = 2$ as $\sigma_1\sigma_2\sigma_2 = (\sigma_1\sigma_2)(\sigma_2)$

Goal: Given $\beta \in B_\Gamma^+$ we will try to read the Garside normal form of β from the action of β on $Com(A_\Gamma - mod)$. From this will follow that the action distinglish positive braids.

1.2.5 Linear complexes of projectives.

Lemma: Up to grading shift the only projective modules over A_Γ are $\{P_i\}_{i\in I}$.

Definition: A complex of projective mod is a complex

$$M^* = \longrightarrow M_k \longrightarrow M_{k-1} \longrightarrow$$

such that it is homotopic to a complex all of whose terms are direct sums of $P_i < k >$.

A complex of projectives M^* is linear if the part of homological degree $k : M_k$ is of the form $\oplus P_i < k >$, that is, the homological degree and the internal grading coincide[6].

1.2.6 Example:

$$\Phi_1(P_2) = P_2 < 0 > \longrightarrow P_1 < 1 >$$

is linear. On the other hand

$$\Phi_1(P_1) = 0 \longrightarrow P_1 < 2 >$$

is not linear.

More generally, we define $T^{\leq k} \subset Com(A_\Gamma)$ to consist of all complexes M^* such that $M_s \simeq \oplus P_i < l >, l - s \leq k$.[7]

Linear complexes from the topological point of view.

Take $\Gamma = A_2 = * - *$

$\otimes$	Garside lenght	P_1	P_2
σ_1	1	$0 \to P_1 < 2 >$	$P_2 \to P_1 < 1 >$
σ_2	1	$P_1 \to P_2 < 1 >$	$0 \to P_2 < 2 >$
$\sigma_1\sigma_2\sigma_1$	1	$P_2 < 3 > [2]$	$P_1 < 3 > [3]$
$\sigma_1\sigma_2$	2	$0 \to P_2$	$0 \to P_2 < 2 > \to P_1 < 3 >$

[6]Given any complex of projectives, it is isomorphic to a minimal complex, that is, the Gauss elimination reduces to a unique complex up to homotopy that doesnot depends on the order of the reductions.

[7]We can think of the corresponding t- structure, $Com(A_\Gamma)$ is triangulated and the Kernel is given by linear complexes.

What is the minimum k such that $\beta \cdot (\oplus_i P_i) \in T^{\leq k}$?

Theorem. $\beta = e(w_k)e(w_{k-1}) \cdots e(w_1) \in B_\Gamma^+$ has Garside lenght k, iff $\beta \cdot (\oplus_i P_i) \in T^{\leq k}$ but $\beta \cdot (\oplus_i P_i) \notin T^{\leq k-1}$.

And to determine $\beta = e(w_k)e(w_{k-1}) \cdots e(w_1)$ the decomposition we need to compute w_k, it sufices to determine $i \in I$ such that: $l(s_i w_k) < l(w_k) \in W_\Gamma$.

1.2.7 Theorem

$$\{i \in I | l(s_i w_k) < l(w_k)\} = \{i \in I | P_i \in T^{=k}(\beta_j \otimes P_j)\}$$

here $P_i \in T^{=k}$ means that $P_i < m + k >$ occurs in homological degree m in a minimal complex for $\beta(\otimes_j P_J)$.

By this theorem, after iteration eventurally we will have a linear complex, and then we also have the corresponding coeficients.

1.2.8 Corollary

The action of the Braid group on $Com(A_\Gamma - mod)$ distinguishes positive braids. That is, if $\beta_1, \beta_2 \in B_\Gamma^+, \beta_1(\otimes P_i) = \beta_2(\otimes P_i)$ iff $\beta_1 = \beta_2$.

1.2.9 Corollary

The map of monoids $\beta_\Gamma^+ \to \beta_\Gamma$ is injective.

As we just checked that the map $B_\Gamma^+ \to B_\Gamma \to End(Com{-}A_\Gamma)$ is injective, so the factors are injective.

1.2.10 Proposition

Suppose Γ is of finte type, ADE, then the action of B_Γ on a set X is faithful iff the restriction to B_Γ^+ is faithful, that is the action distinguish positive braids. Proof: We'll show that any braid $\beta \in B_\Gamma$ can be writen as $\beta_+\beta_-, \beta_+ \in B_\Gamma^+, \beta_- \in (B_\Gamma^+)^{-1}$.

This requires a review.

To see this, let $\triangle = e(w_0), w_0$ the longest element of W_Γ, then [8]

- $\sigma_i^{-1}\triangle \in B_\Gamma^+$,
- $\triangle\sigma_i\triangle^{-1} \in B_\Gamma^+$.

[8] for example, for $\Gamma = A_n, \triangle = e_1(e_2e_1)(e_3e_2e_1) \cdots (e_{n-1}e_{n-2} \cdots e_2e_1)$

then in a decomposition $\beta = \sigma_{i_1}^{\epsilon_{i_1}} \sigma_{i_2}^{\epsilon_{i_2}} \cdots \sigma_{i_m}^{\epsilon_{i_m}}$ replace any appearance of σ^{-1} by $\triangle^{-1}\alpha, \alpha \in B^+\_\Gamma$ and get a relation of the form $\beta = \sigma_{i_1} \cdots \triangle^{-1} \cdots \sigma_{i_k}$ and move all $\triangle^{-1}$ to the right to obtain the desired decomposition.

1.2.11 Corollary

The action of β_Γ on $Com(A_\Gamma - mod)$ is faithful when Γ is of type ADE.

Remark. The faithfulness result implies other such results for several other categorial actions. Notably: Rouquiers action.